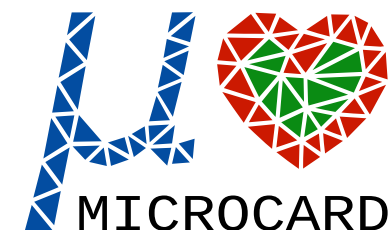
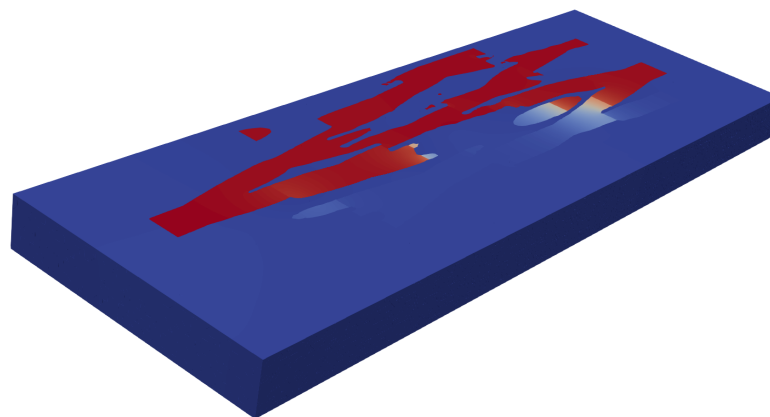
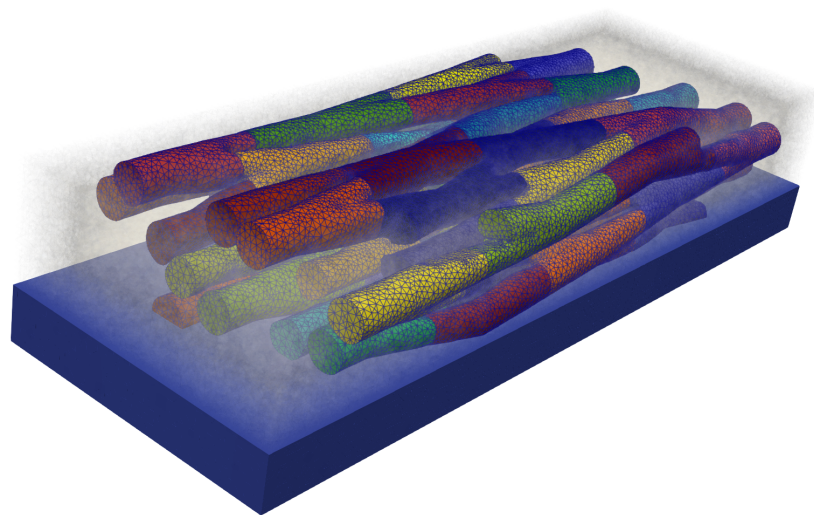


BDDC with Algebraic Adaptivity and Compressed Communication for Cardiac Electrophysiology

F.Chegini, T. Steinke, M. Weiser



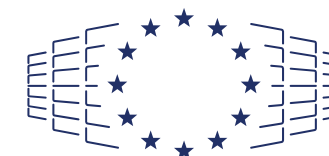
With funding from the:



Federal Ministry
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and Space



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the European Union

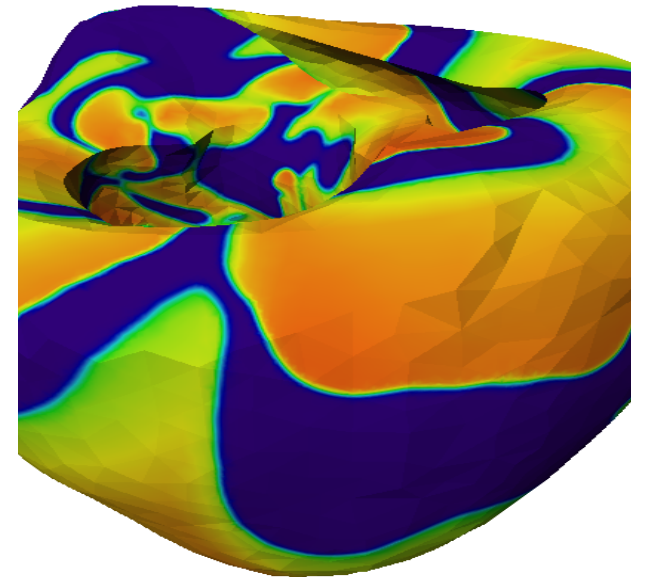


EuroHPC
Joint Undertaking

This work was supported by the MICROCARD-2 project (project ID 101172576). The project is supported by the EuroHPC Joint Undertaking and its members (including top-up funding by ANR, BMBF, and Ministero dello sviluppo economico). Funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or EuroHPC. Neither the European Union nor EuroHPC can be held responsible for them.

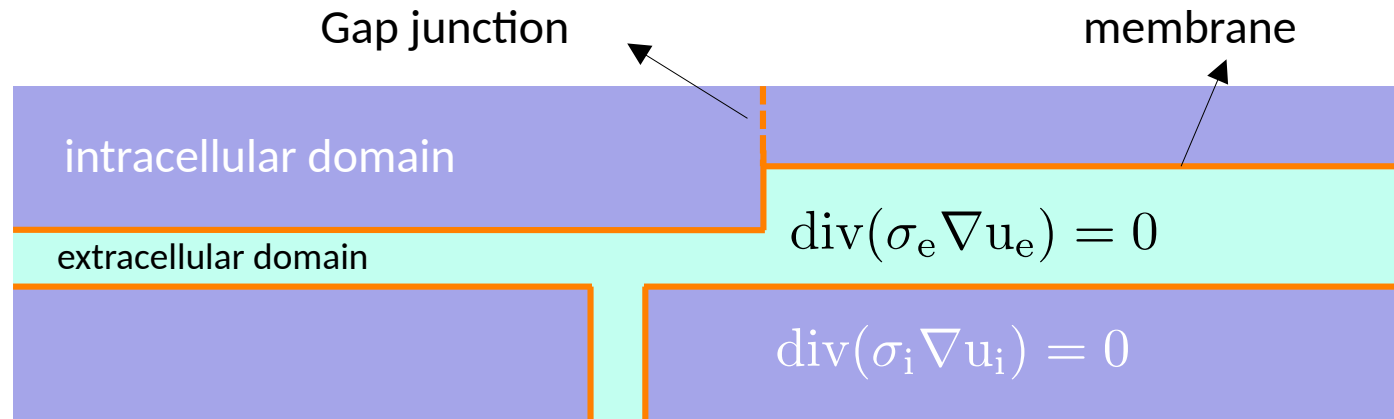
Why EMI Simulation

- EMI consists of intracellular, extracellular, and membrane domains.
- Cardiac tissue heterogeneity at the cellular scale affects action-potential propagation
- This level of fidelity is relevant for **arrhythmia**, **scarring**, and damaged tissue.



Accurate cellular-scale models are too **expensive** to run routinely without aggressive algorithmic acceleration.

EMI Equation



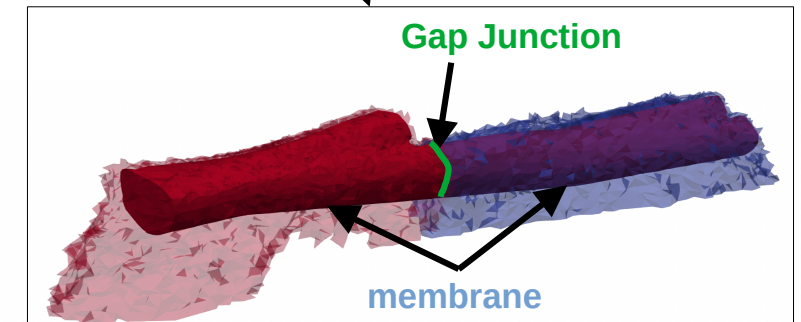
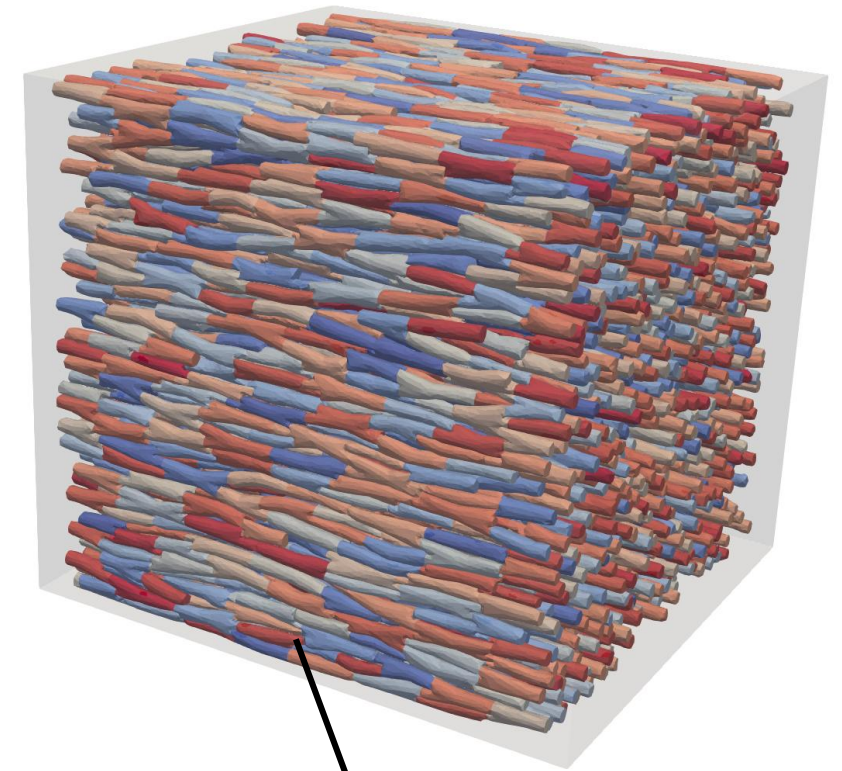
$$-\nabla \cdot (\sigma_i \nabla u_i) = 0 \quad \text{in } \Omega_i$$

$$-\nabla \cdot (\sigma_e \nabla u_e) = 0 \quad \text{in } \Omega_e$$

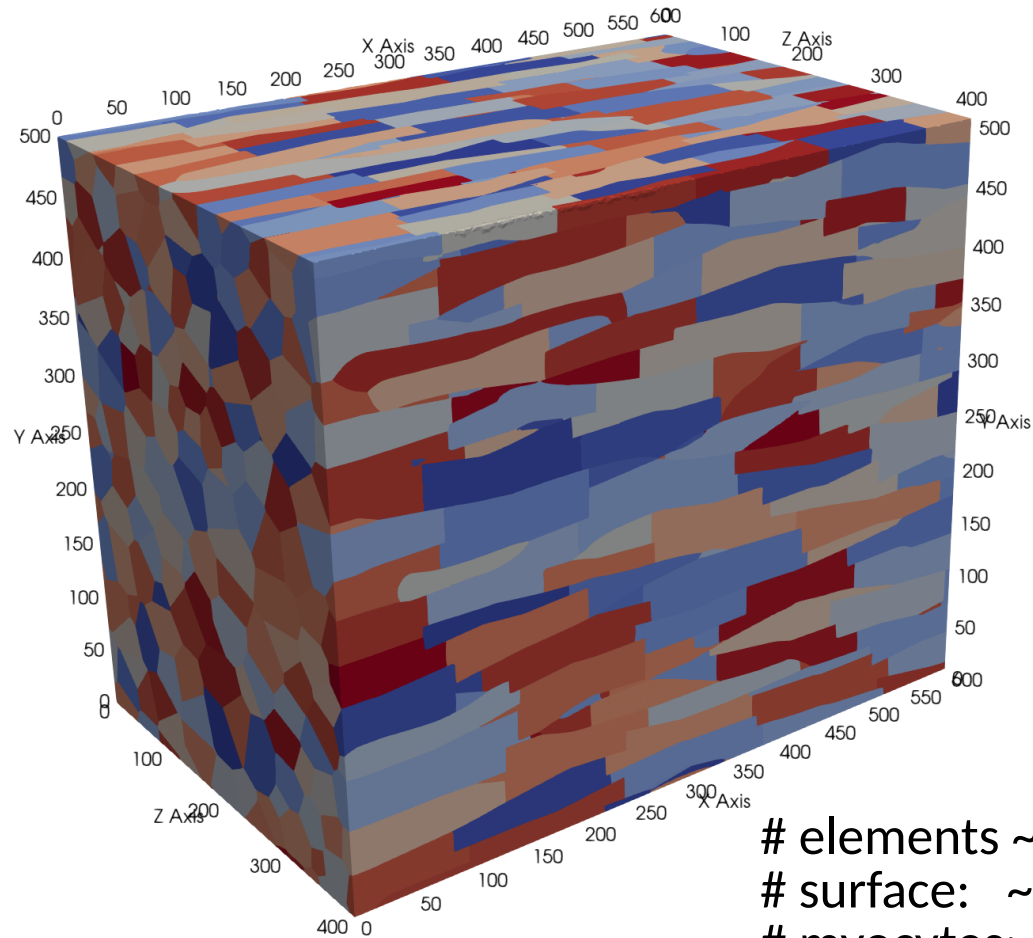
$$u_i - u_e = v_m \quad \text{on } \Gamma_m$$

$$n_i \cdot \sigma_i \nabla u_i = -n_e \cdot \sigma_e \nabla u_e = I_m$$

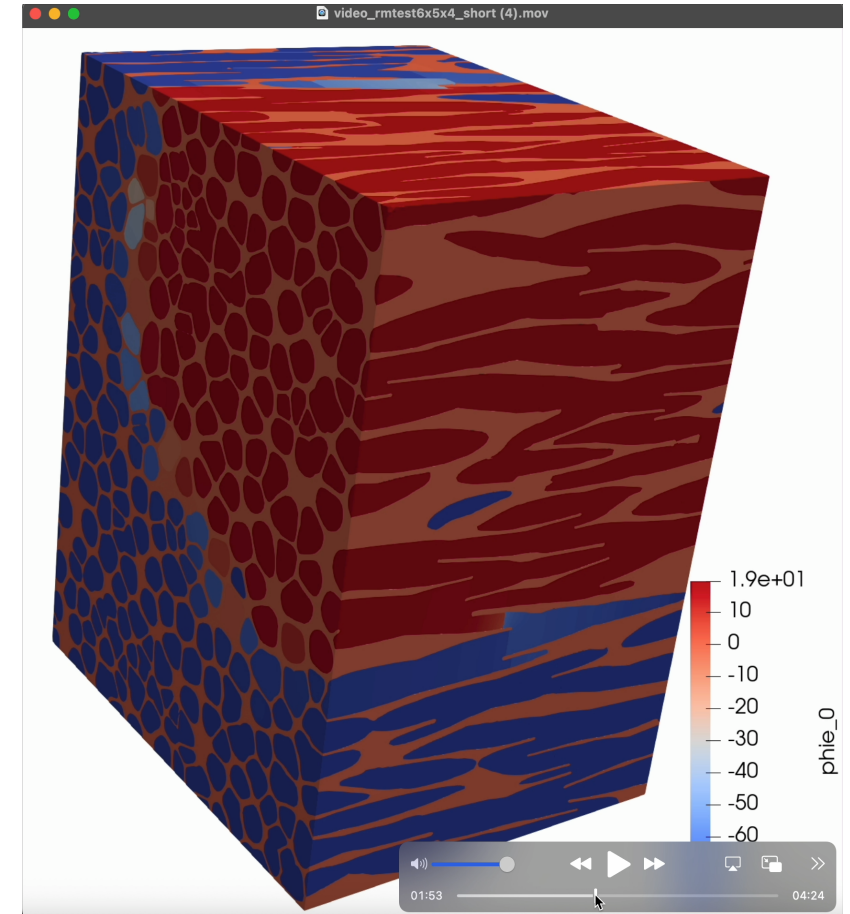
$$C_m \partial_t v_m + I_{\text{ion}}(v_m, \dots) = I_m$$



EMI Simulation



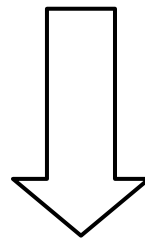
elements ~20 millions
surface: ~4 millions
myocytes: 1188
extra: 1227
dim: 600x500x400 μ m



Cellular-scale EMI simulations: **expensive** due to *fine spatial structure* with *stiff time dynamics*.

SDC as higher-order time integration: **practical** combined with **Algebraic Adaptivity(AA)**

BDDC preconditioner: **scalable** + **compressed communication** to **reduce** interface-exchange without harming convergence



The combination improves **accuracy + efficiency** preserving robustness for large EMI simulations.

Accuracy vs efficiency

$$M\dot{z} = L(z)$$

Challenge

- spatially local structures → fine grids
- fast dynamics → small time steps



huge
computational effort

Approaches

- massive parallelization, space-time discretization
- model selection & coupling
- mesh & time step adaptivity



Classical adaptivity
fails to
provide high efficiency

Wish list

- **SDC + Adaptive coarsening** to the **algebraic level**
 - Avoid mesh modification & reassembly
 - Reduced the computational time
- Leveraging **domain decomposition** method



SDC method + **Algebraic Adaptivity(AA)**



BDDC + Data compression + SDC + AA

Spectral deferred correction method(SDC)

Problem

$$M\dot{z} = L(z)$$

Time grid

$$0 = \tau_0 < \tau_1 < \dots < \tau_m = \tau$$

approximate rhs

$$\tilde{z} \in \mathbb{P}_m, z_i = z(\tau_i)$$

$$M(z_{i+1} - z_i) = \int_{\tau=\tau_i}^{\tau_{i+1}} L(z) d\tau, \quad i = 0, \dots, m-1$$

$$\approx \sum_{j=0}^m S_{ij} L(z_j).$$

Assemble once

$$M(\delta z_{i+1}^k - \delta z_i^k) - \sum_{j=0}^n S_{ij} L'(z_j^k) \delta z_j^k$$

$$= -M(z_{i+1}^k - z_i^k) + \sum_{j=0}^n S_{ij} L(z_j^k)$$

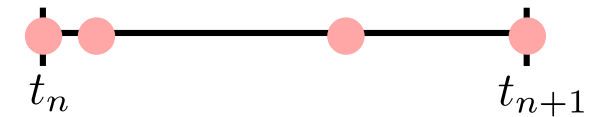
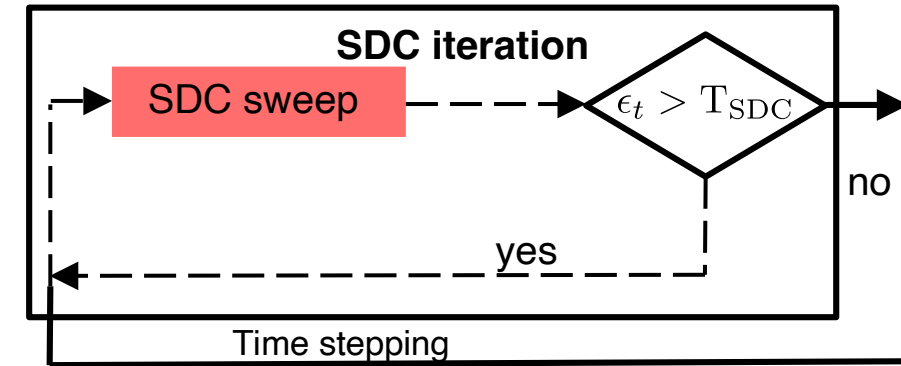
Assemble at each collocation point

Collocation points

$$\tau_0, \dots, \tau_{m-1}$$

$$z^{k+1} = z^k + \delta z^k$$

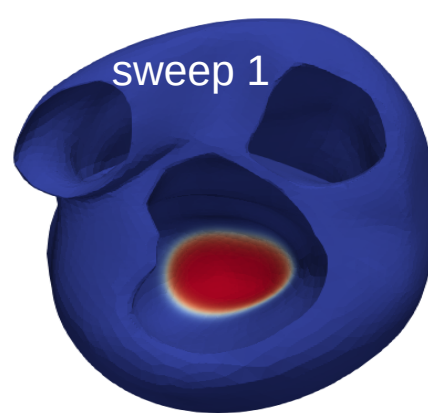
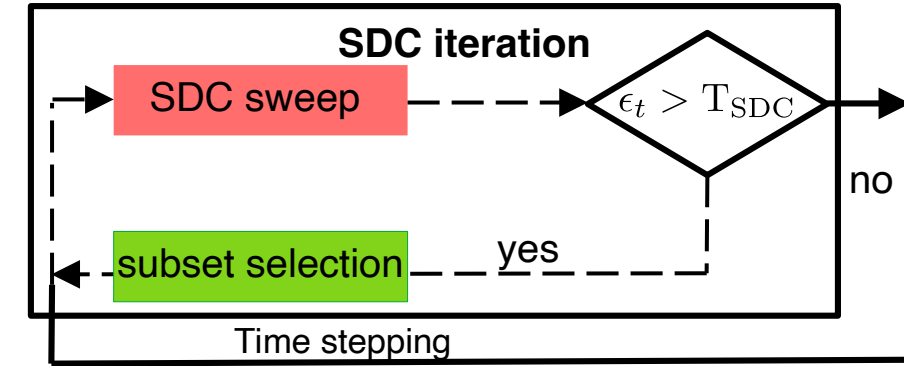
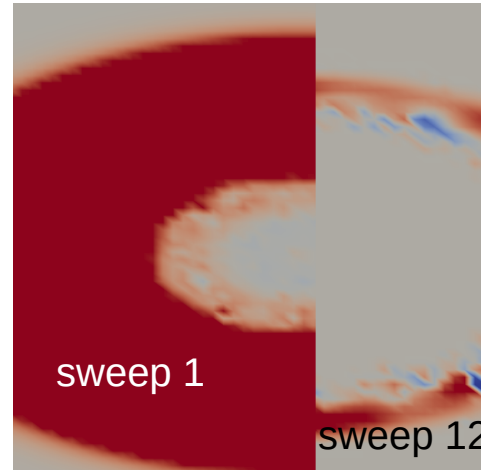
high-order quadrature on a collocation time grid



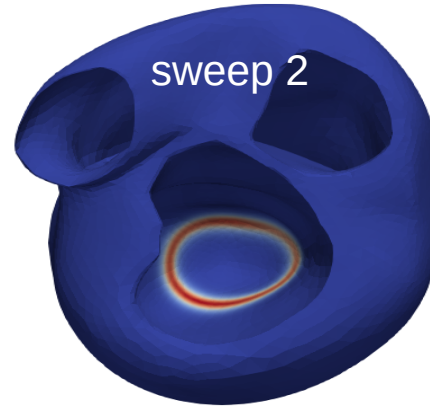
Example: Algebraic Adaptivity (AA)

limit SDC improvements to spatial regions with significant dynamics by selecting DoFs

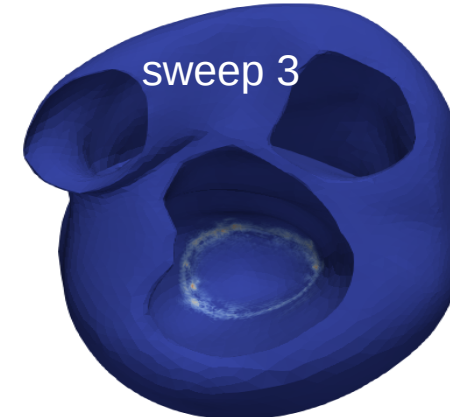
impose homogeneous Dirichlet b.c. for SDC corrections



~600k



~3k

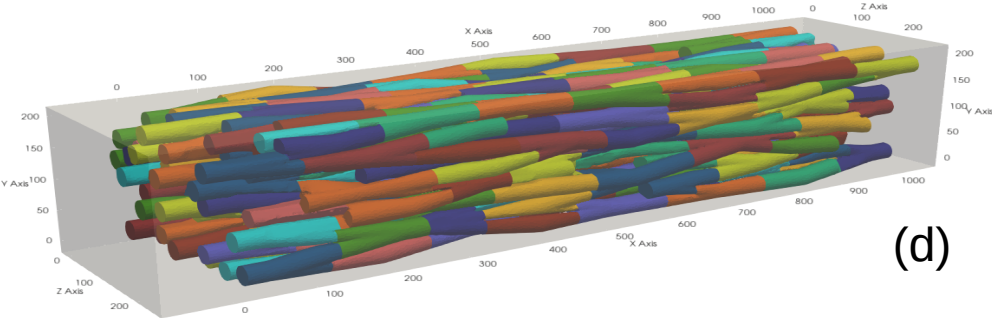
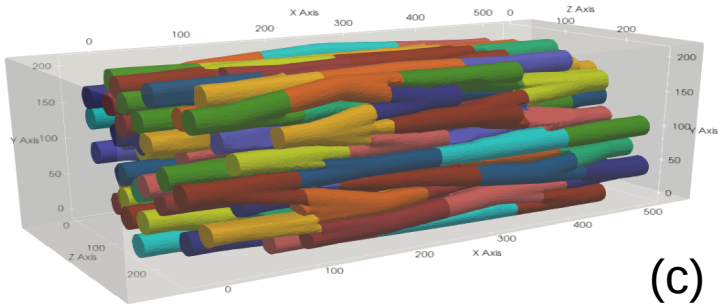
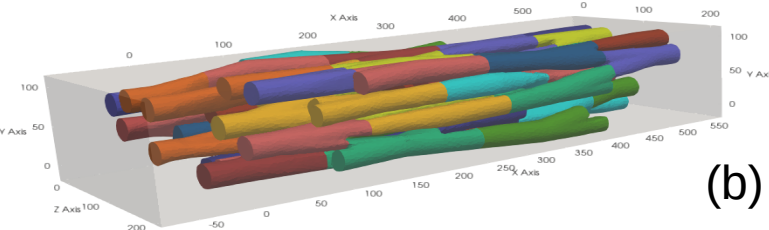
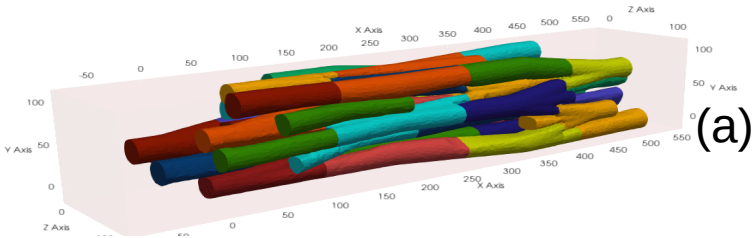


0.4k

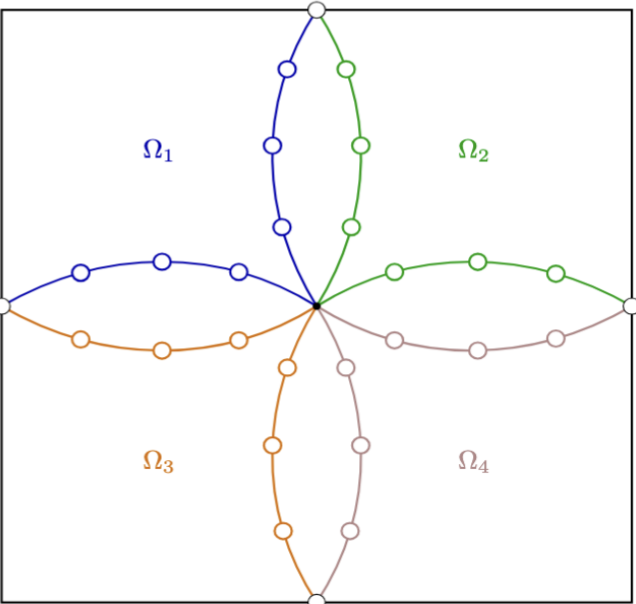
[Chegini, Weiser, et al. 2023] Efficient adaptivity for simulating cardiac electrophysiology with spectral deferred correction methods

Speedup on 3D EMI Models with SDC + AA

mesh	vertices	myocytes	dofs	SDC+AA
(a)	332,179	44	394,106	4.526
(b)	552,489	86	628,586	4.753
(c)	1,314,675	208	1,664,019	5.432
(d)	2,521,502	415	3,286,868	5.863

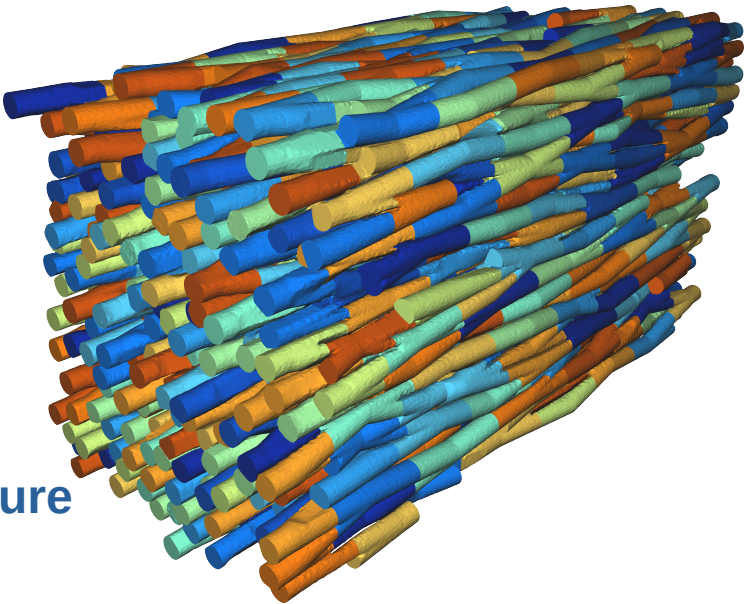


BDDC preconditioner



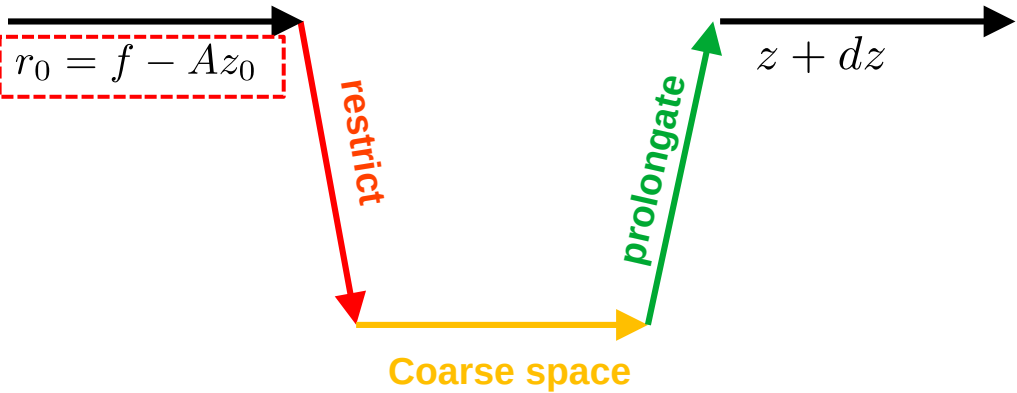
BDDC

- Subdomain-local solves
- Interface correction
- Data compression



Block-diagonal structure

$$\begin{bmatrix} A_0 & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{bmatrix} \in \mathbb{R}^N$$



dofs	CG-noPrec			itr	BDDC	
	itr	CPU time(s)	k_2		CPU time(s)	k_2
33,698	1111	6.595	24,390	11	0.105	52
75,122	2388	58.711	83,610	12	0.307	134
132,930	2500	69.238	32,560	12	0.630	157
207,122	5209	608.205	138,080	10	0.771	229
297,698	4606	400.290	305,810	10	1.245	196

SDC + BDDC + AA

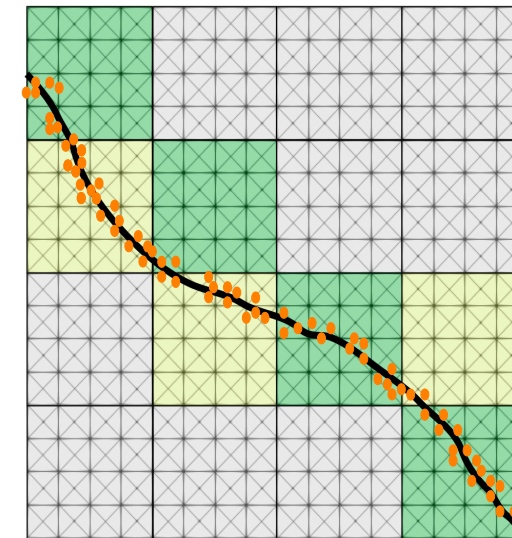
Challenges

- variable step sizes for IMEX Euler in Radau grids
- changing system size due to adaptivity

Subdomain-wise subset selection in BDDC

Algebraic adaptivity

- subdomain-wise selection of dofs
 - keep factorization for subdomains
 - global Schur complement system
 - factorize again
 - add constraints & Krylov for Schur² system
- in both cases: careful averaging of residuals needed (nullspace in EMI)

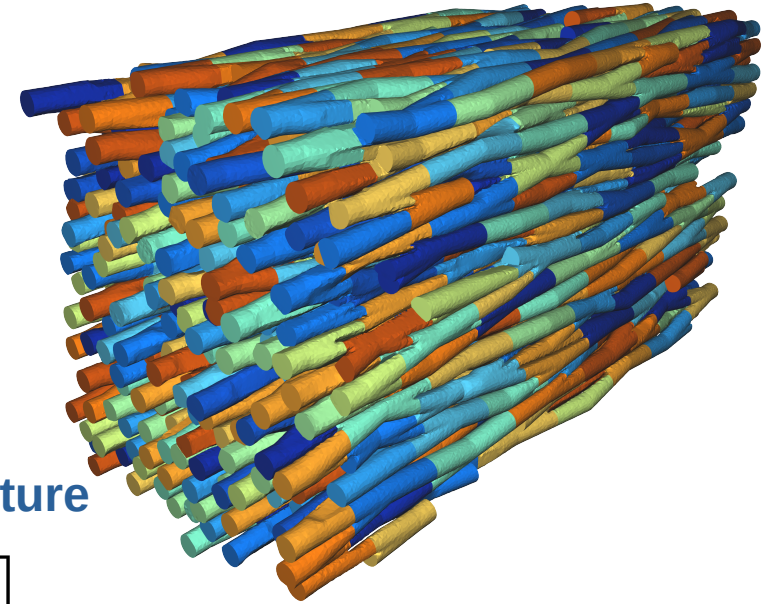
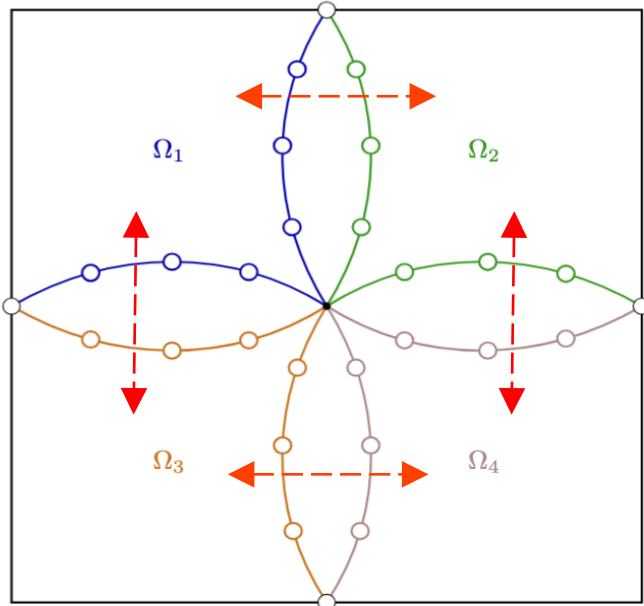


- SDC methods are **highly flexible** due to their corrective structure
- they play well together with all kinds of **inexactness**
- this makes them highly attractive for **efficient PDE solvers**

BDDC preconditioner

BDDC

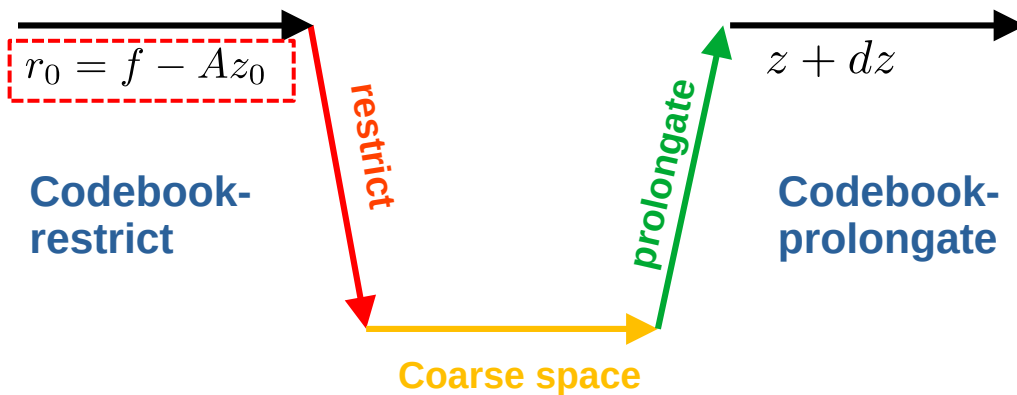
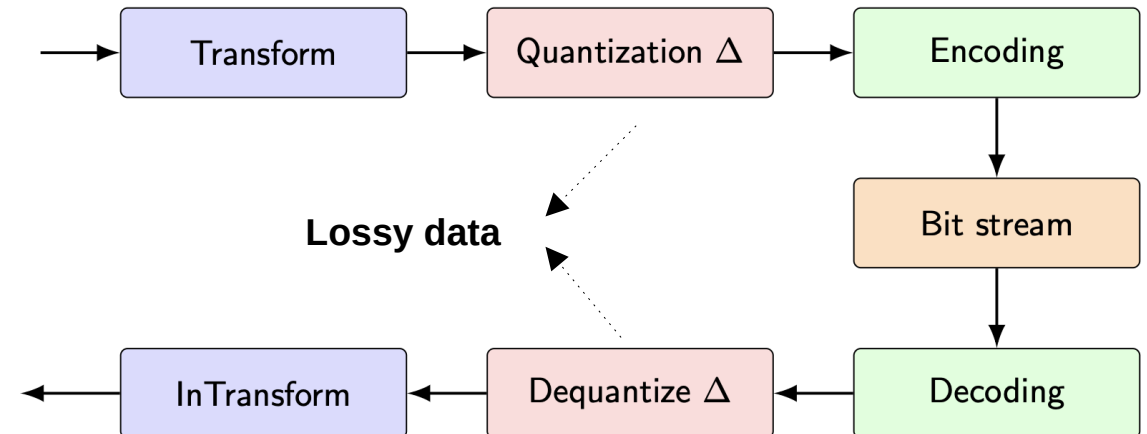
- Subdomain-local solves
- Interface correction
- Data compression



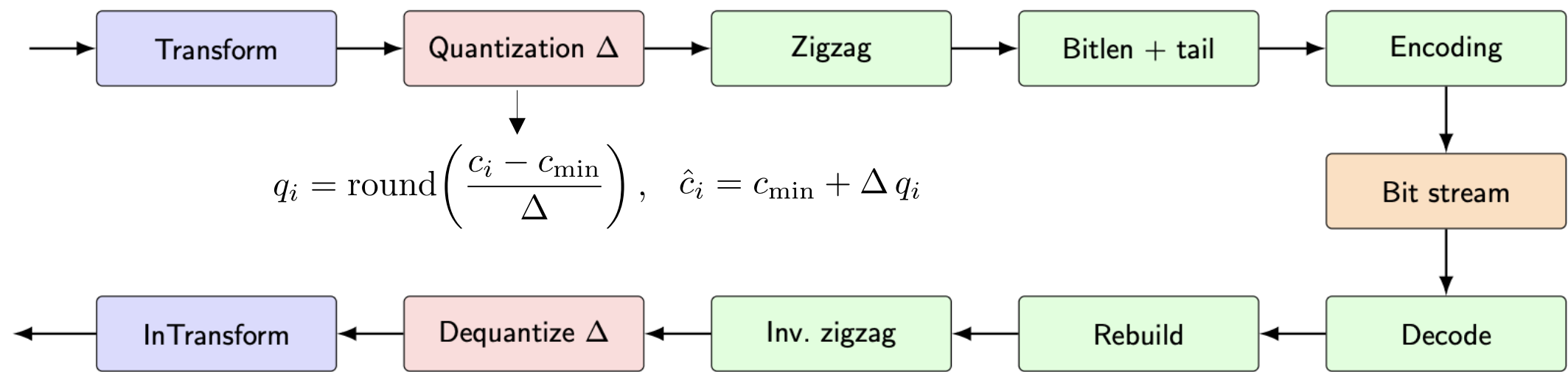
Block-diagonal structure

$$\begin{bmatrix} A_0 & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{bmatrix} \in \mathbb{R}^N$$

Data compression

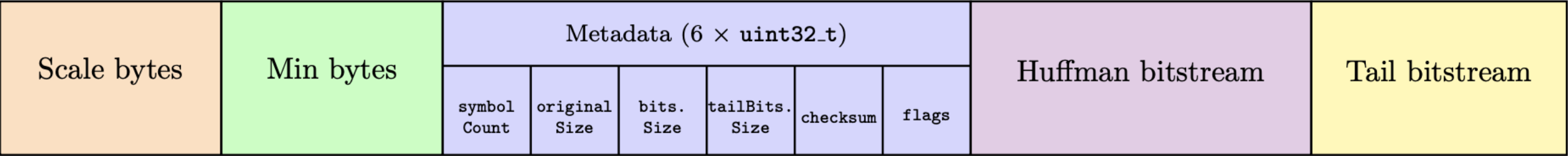


Dara compression + Bitstream

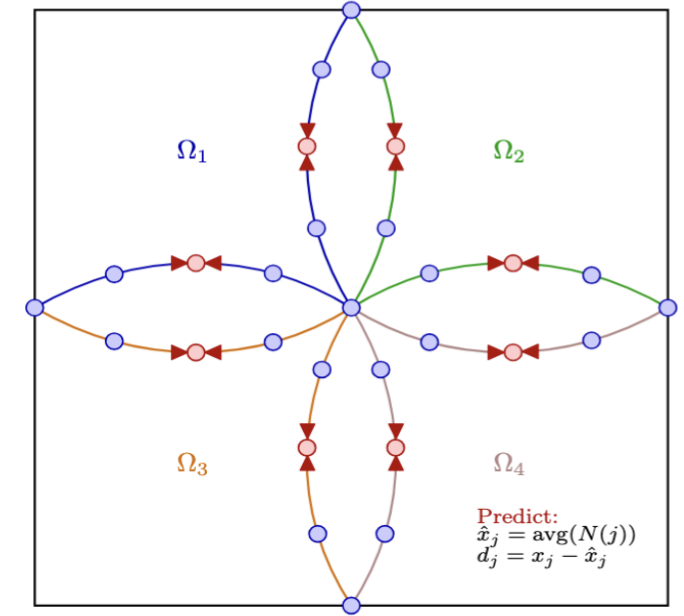
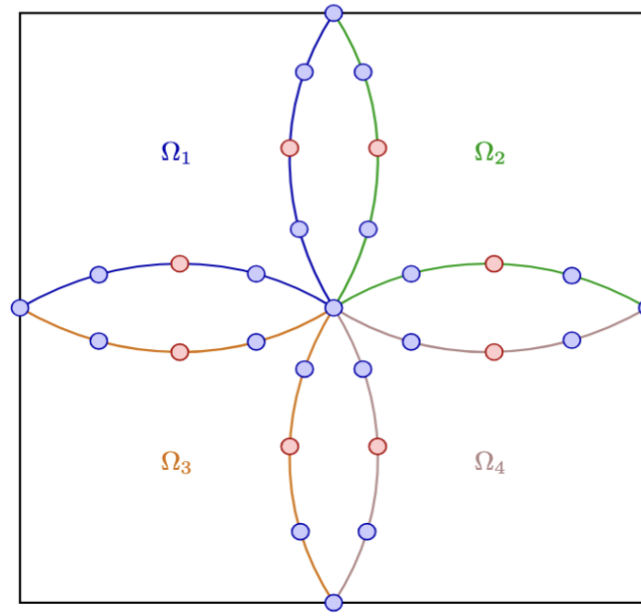
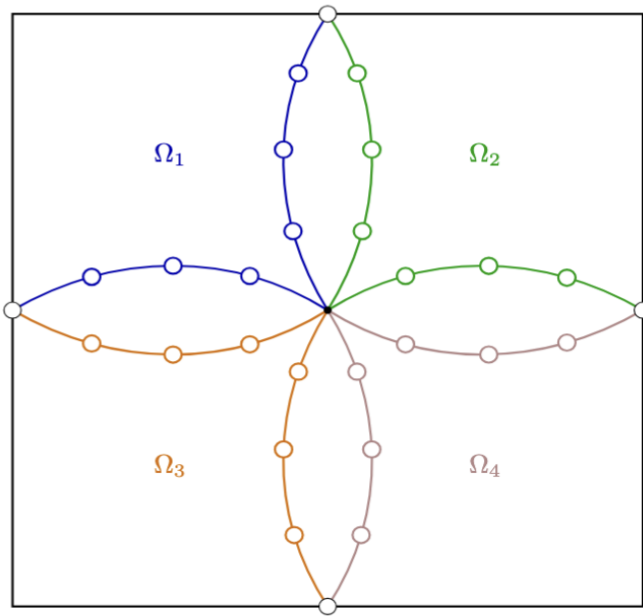


Transform: lifting-scheme wavelets
Encoding: Huffman coding

Bitstream included metadata



Transform: Graph Lifting method



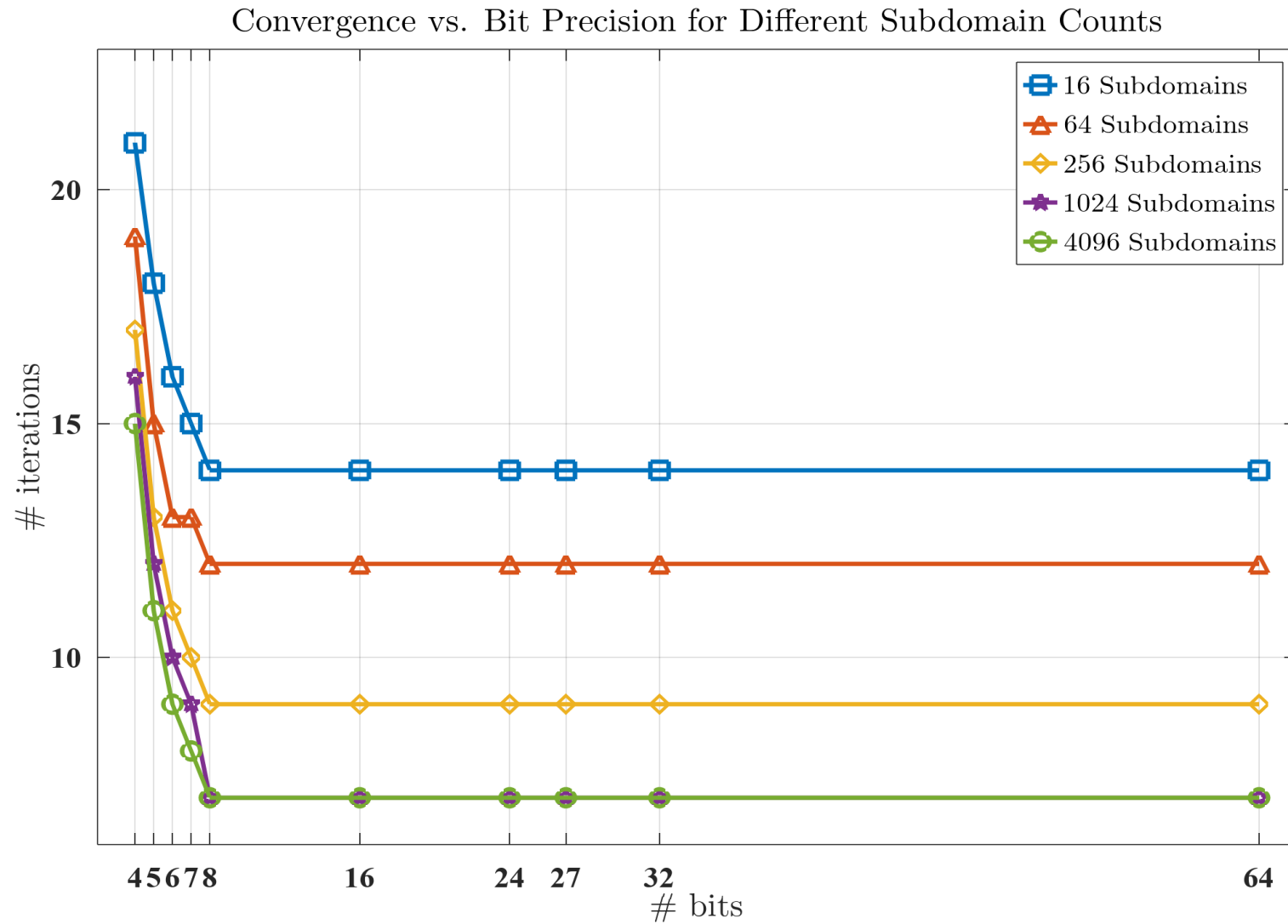
BFS bipartition on the interface graph

- Build a graph where each interface DoFs is a node, and edges neighbor interface coupled DoFs.
- Run BFS (Breath- First Search) on each connected component
- Color 0/1 (0: coarse, 1: fine)

Graph lifting:

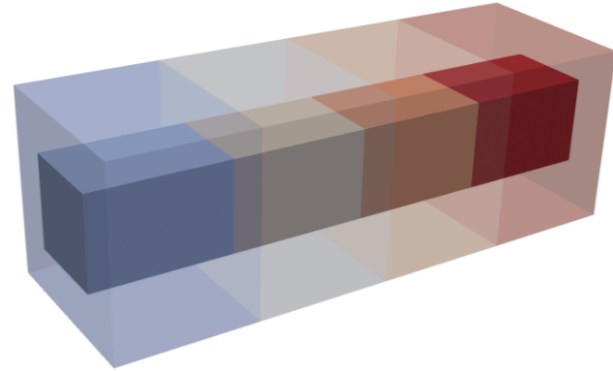
$$p = \frac{\sum (w_i * x_{ci})}{\sum (w_i)}$$

Impact of quantization



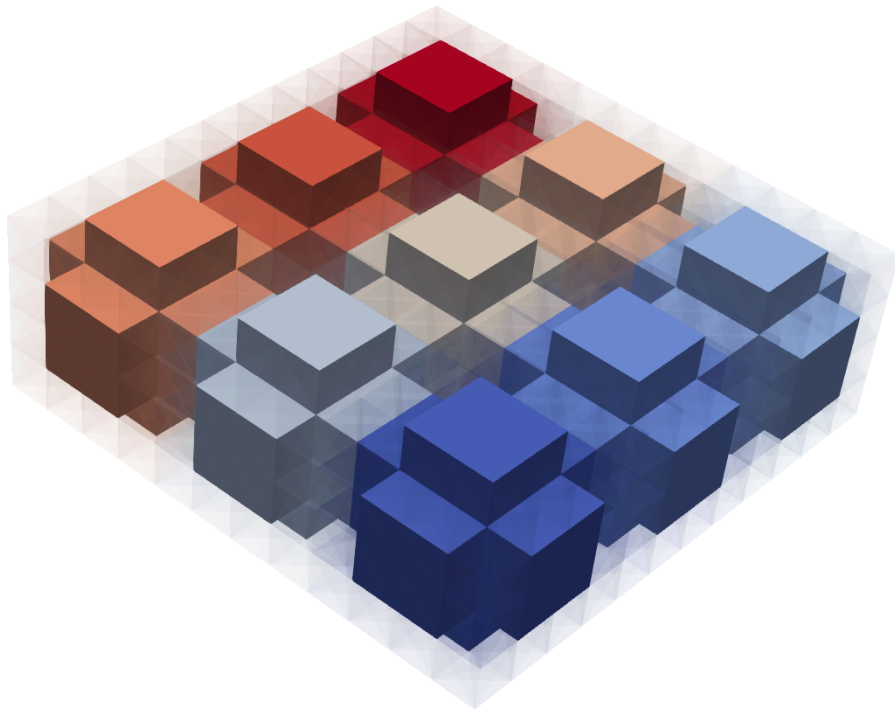
no transform, uniform scalar quantization, 2D/3D Poisson

Strong and weak Scaling with quantization



strong Scaling

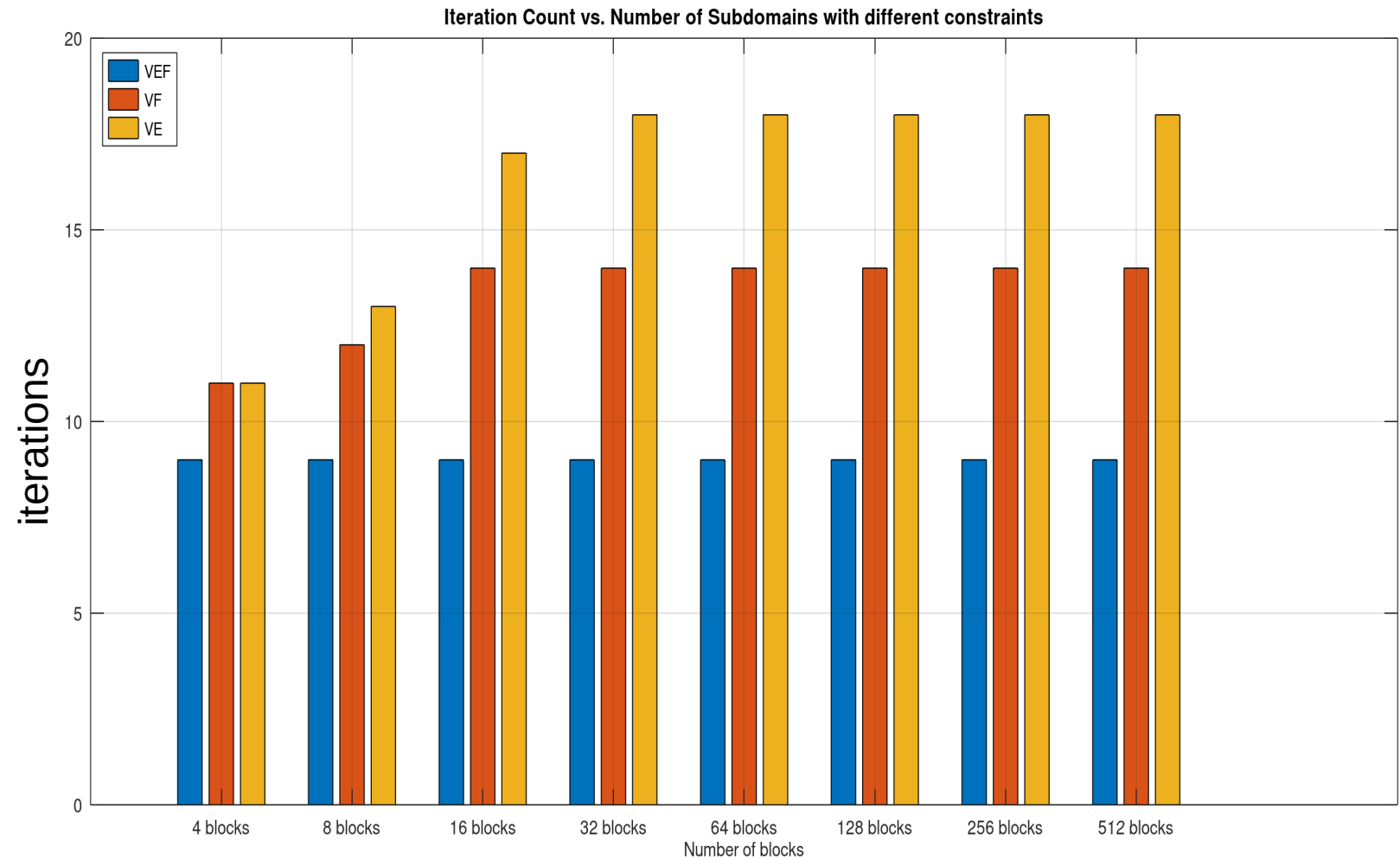
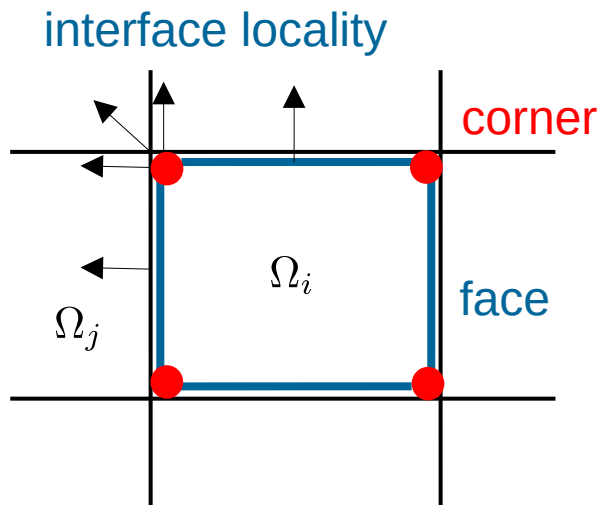
#refine	#subdomains	it	κ	min #dofs	max #dofs
0	8	9	1.59803	77	128
1	8	15	2.37519	300	580
2	8	17	2.64967	1430	3176
3	8	17	2.49507	8106	20176



weak Scaling

#blocks	#subdomains	64 bits		24 bits	
		it	κ	it	κ
3x3x1	18	10	1.77782	10	1.77782
3x3x2	36	10	1.81586	10	1.81585
3x3x3	54	10	1.77251	10	1.77249
4x4x4	128	10	1.73902	10	1.75973
5x5x5	250	10	1.73902	10	1.75973
6x6x6	432	10	1.73902	10	1.75973

Numerical Results: Weak Scaling with quantization



no transform, uniform scalar quantization, 3D EMI
convergence unaffected for quantization 24-64 bits

problem size

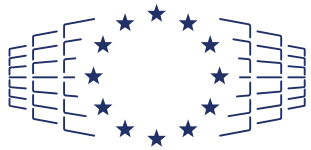
SDC + BDDC + Data compression

refine	BDDC+AA	BDDC+AA+bits
0	1.510	1.818
1	1.785	2.243
2	1.915	2.468

Compression ratio in SDC + BDDC
(32 → 20 bits)

Conclusion

- **SDC** provides a flexible high-order framework for EMI time integration.
- **Algebraic adaptivity** makes that framework affordable by reducing local work.
- SDC + Applied nested sub-matrices for **BDDC** preconditioner + **Data compression**
 - Load imbalance in distributed simulations of large scale problems
 - + Reducing the energy consumption of large scale simulations.



EuroHPC
Joint Undertaking

With funding from the:



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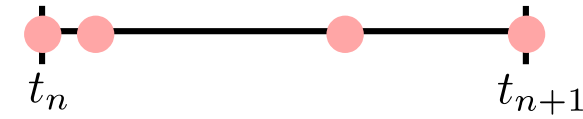
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Thanks for your attention!



SDC + (AA)

m Collocation points



sweeps



SDC + Algebraic adaptivity

Extracted Nested sub-matrices



Algebraic adaptivity

Higher order with
degree of freedom subset selection

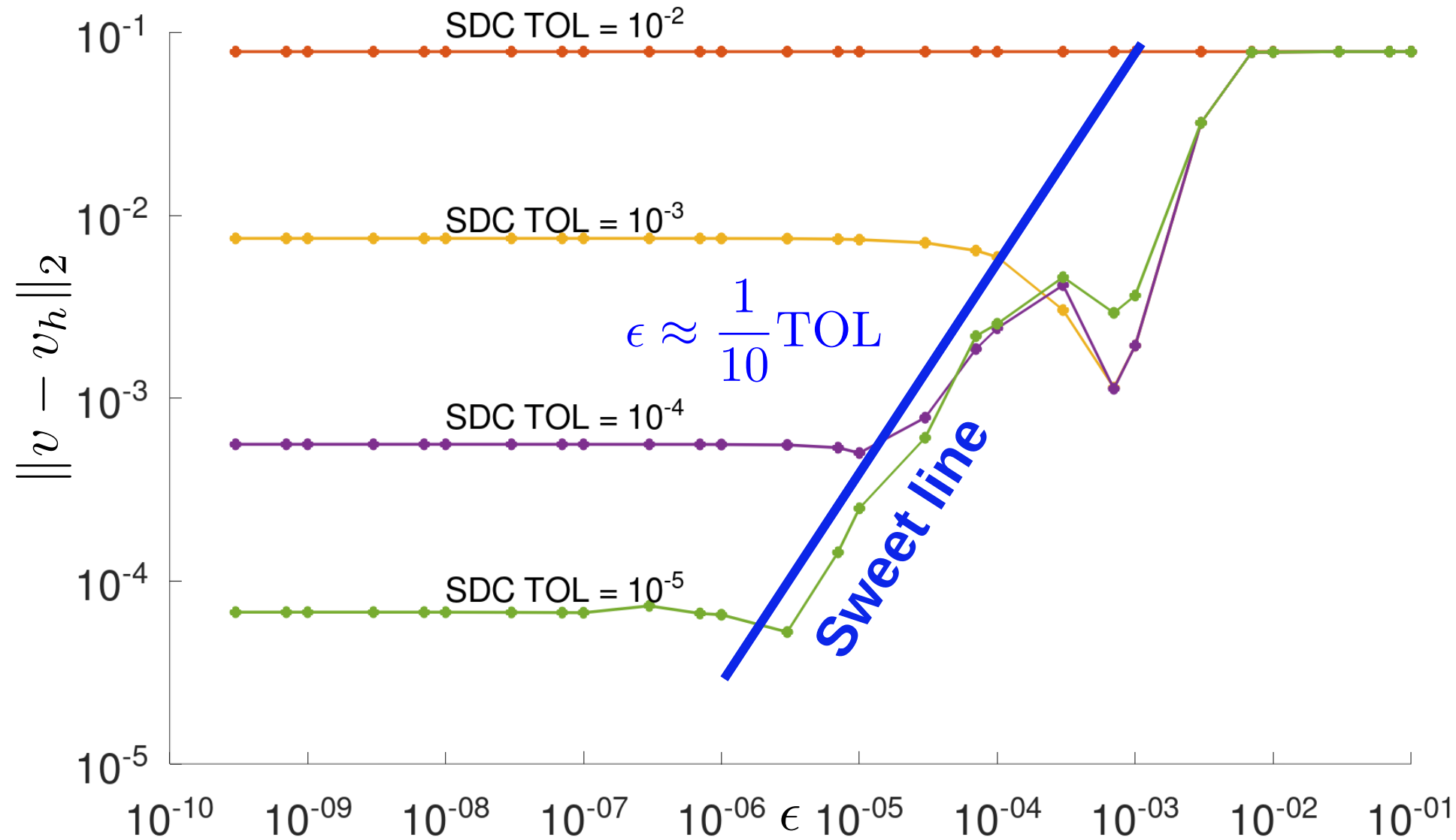
Realization

- cheap submatrix extraction
- nested approximation spaces

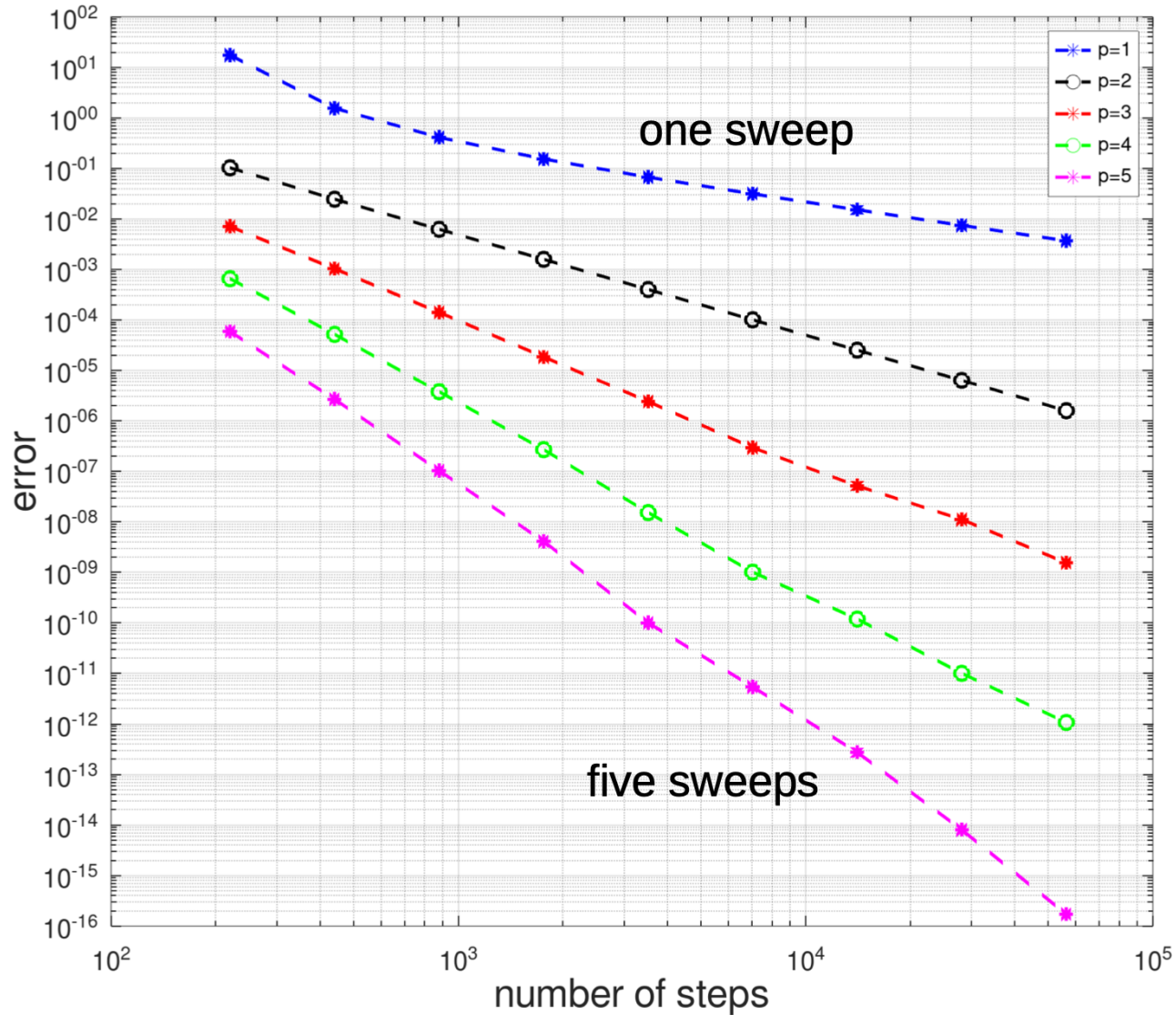
+ BDDC preconditioner

Applied nested sub-matrices for preconditioner

Drop Tolerant in SDC



SDC convergence



cell-by-cell model 2D
220-56320 time steps
Radau-II(3) collocation
L2 norm of final time error